Constrained Optimization : Step by step

Maximizing subject to a set of constraints

Problem:

$$\text{Max } U = A^{1/2} B^{1/2}, \quad P_A = \$4, \quad P_B = \$2, \quad I = \$120$$

s.t

$$4A + 2B = 120$$

$$\mathcal{L} = \underbrace{A^{1/2} B^{1/2}}_{f(A,B)} + \lambda\big(\underbrace{120 - 4A - 2B}_{g(A,B)}\big)$$ => We can write out the lagrangian (Step 1)

$$\frac{\partial \mathcal{L}}{\partial A} = 1/2\, A^{-1/2} B^{1/2} - 4\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial B} = 1/2\, B^{-1/2} A^{1/2} - 2\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 120 - 4A - 2B = 0$$

Take the partial derivative with respect to each variable (Step 2)

$$\lambda = \frac{\cancel{1/2}\, A^{-1/2} B^{1/2}}{\cancel{4}\ 2} = \frac{\cancel{1/2}\, B^{-1/2} A^{1/2}}{\cancel{2}}$$

Solve the first order conditions for Lambda (step 3)

$$A^{-1/2} B^{1/2} = 2\, B^{-1/2} A^{1/2}$$

$$\frac{B^{1/2}}{2 \cdot B^{-1/2}} = \frac{A^{1/2}}{A^{-1/2}}$$

$$\frac{B}{2} = A \Rightarrow \boxed{B = 2A}$$

Set the two expressions for lambda equal to each other (step 4)

$$4A + 2B = 120$$

$$4A + 2(2A) = 120$$

$$8A = 120$$

$$\boxed{A^* = 15} \quad \boxed{B^* = 2 \cdot 15 = 30}$$

Use the constraint to solve for the two variables separately (Step 5)

check:

$$4 \cdot (15) + 2(30) = 60 + 60 = 120 \checkmark$$

Minimizing subject to a set of constraints

Problem:

Firm output $= Q = K^{1/2} L^{1/2}$

$P_K = 3$, $P_L = 9$. The firm wants to minimize the total costs of producing at least 100 units of output.

$$\left.\begin{array}{l} \min\ 3K + 9L \\ \quad s.t \\ K^{1/2}L^{1/2} \geq 100 \end{array}\right\} \text{The problem we face is}$$

We can write the problem as

$$\mathcal{L} = 3K + 9L - \lambda\left(K^{1/2}L^{1/2} - 100\right) \quad (\text{step } 1)$$

$$\left.\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial K} = 3 - \frac{1}{2}K^{-1/2}L^{1/2}\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial L} = 9 - \frac{1}{2}L^{-1/2}K^{1/2}\lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = -\left(K^{1/2}L^{1/2} - 100\right) = 0 \end{array}\right\} \text{Take the partial derivative w.r.t. each variable (step 2)}$$

Solve the first order conditions for lambda. (step 3)

$$\lambda = \frac{\cancel{3}}{\cancel{\frac{1}{2}}K^{-1/2}L^{1/2}} = \frac{\cancel{9}\ 3}{\cancel{\frac{1}{2}}L^{-1/2}K^{1/2}}$$

$$\left.\frac{\cancel{L^{-1/2}}K^{1/2}}{\cancel{L^{-1/2}}K^{-1/2}} = \frac{3\cancel{K^{-1/2}}L^{1/2}}{L^{-1/2}\cancel{K^{-1/2}}}\right\} \text{set the two expressions for lambda equal to each other (step 4)}$$

$$\boxed{K = 3L}$$

$$\left.\begin{array}{l} K^{1/2}L^{1/2} = 100 \\ (3L)^{1/2}L^{1/2} = 100 \\ \sqrt{3}\cdot L^{1/2}\cdot L^{1/2} = 100 \\ \boxed{L = \frac{100}{\sqrt{3}} = 57{,}8} \quad \boxed{K = \frac{3\cdot 100}{\sqrt{3}} = 173{,}41} \end{array}\right\} \text{Use the constraint to solve for the two variables separately. (step 5)}$$